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On a system of equations arising in viscoelasticity theory of fractional type

Teodor M. Atanackovic*, Stevan Pilipovic† and Dusan Zorica‡

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Abstract

We study a system of partial differential equations with integer and fractional derivatives arising in the study of forced oscillatory motion of a viscoelastic rod. We propose a new approach considering a quotient of relations appearing in the constitutive equation instead the constitutive equation itself. Both, a rod and a body are assumed to have finite mass. The motion of a body is assumed to be translatory. Existence and uniqueness for the corresponding initial-boundary value problem is proved within the spaces of functions and distributions.

Keywords: fractional derivative, distributed-order fractional derivative, fractional viscoelastic material, forced oscillations of a rod, forced oscillations of a body

1 Introduction

In this paper we study (1) - (4) derived in [8]. The system corresponds to a motion of a viscoelastic rod fixed at one end and of a body of finite mass attached to the other end. Also, an outer force, having the action line coinciding with the axis of the rod, acts at the body attached to the free end of a rod. In the dimensionless form the system of equations, initial and boundary conditions, describing such a motion, reads

$$\frac{\partial}{\partial x}\sigma\left(x,t\right)=\kappa^{2}\frac{\partial^{2}}{\partial t^{2}}u\left(x,t\right),\quad \varepsilon\left(x,t\right)=\frac{\partial}{\partial x}u\left(x,t\right),\ \ x\in\left[0,1\right],\ t>0, \tag{1}$$

$$\int_{0}^{1}\phi_{\sigma}\left(\gamma\right){}_{0}\mathrm{D}_{t}^{\gamma}\sigma\left(x,t\right)\mathrm{d}\gamma=\int_{0}^{1}\phi_{\varepsilon}\left(\gamma\right){}_{0}\mathrm{D}_{t}^{\gamma}\varepsilon\left(x,t\right)\mathrm{d}\gamma,\ \ x\in\left[0,1\right],\ t>0, \tag{2}$$

$$u\left(x,0\right)=0,\quad \frac{\partial}{\partial t}u\left(x,0\right)=0,\quad \sigma\left(x,0\right)=0,\quad \varepsilon\left(x,0\right)=0,\ \ x\in\left[0,1\right], \tag{3}$$

$$u\left(0,t\right)=0,\quad -\sigma\left(1,t\right)+F\left(t\right)=\frac{\partial^{2}}{\partial t^{2}}u\left(1,t\right),\ \ t>0. \tag{4}$$

We note that $(1)_1$ represents equation of motion for an arbitrary material point of a rod. In it, σ denotes the stress at the point x at time t, u is the displacement, κ is a constant representing

*Department of Mechanics, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovica, 6, 21000 Novi Sad, Serbia, atanackovic@uns.ac.rs

†Department of Mathematics, Faculty of Natural Sciences and Mathematics, University of Novi Sad, Trg D. Obradovica, 4, 21000 Novi Sad, Serbia, stevan.pilipovic@dmi.uns.ac.rs

‡Mathematical Institute, Serbian Academy of Sciences and Arts, Kneza Mihaila 36, 11000 Beograd, Serbia, dusan_zorica@mi.sanu.ac.rs

the ratio between the masses of a rod and a body. In $(1)_2$ we use ε to denote the axial strain of a rod, while in (2) ϕ_σ and ϕ_ε denote constitutive functions or distributions, that are assumed to be known. The operator of the left Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$ ${}_0\mathrm{D}_t^\gamma$ is defined as

$$ {}_0\mathrm{D}_t^\gamma y(t) := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{-\gamma}}{\Gamma(1-\gamma)} * y(t)\right), \quad t > 0, $$

where Γ is the Euler gamma function, $*$ is a convolution, i.e., if $f, g \in L^1_{loc}(\mathbb{R})$, $\operatorname{supp} f, g \subset [0,\infty)$, then $(f * g)(t) := \int_0^t f(\tau)\, g(t-\tau)\, \mathrm{d}\tau$, $t \in \mathbb{R}$. We refer to [9, 13, 16] for a detailed account on fractional calculus. In (3) the initial conditions for an initially undeformed rod are presented. Finally, (4) represents the boundary conditions corresponding to a rod with one end fixed at $x = 0$ and with force F acting at the body attached to the other end (at $x = 1$). Note that initial-boundary value problem (1) - (4) represents a generalization of a problem of forced oscillations in the case of a light rod, presented in [4]. We refer to [8] for the details for the physical interpretation basis of system (1) - (4).

The main novelty of our approach is that we consider a constitutive equation in essentially new way. Instead of examining operators acting on stress and strain in constitutive equation separately, we analyze their quotient after the application of the Laplace transform to (2), that we denote by M, see (9). In this context M appears as a new important quantity which reflects the inherent properties of a material of a rod. General form of M requires detailed mathematical analysis which is given in the paper.

The aim of this paper is to prove the existence and uniqueness of a solution to system (1) - (4). Our main results are stated as Theorems 1 and 2 bellow. In proving these theorems we use several auxiliary results presented in a separate section.

Constitutive equations (2) were used earlier in [2, 6, 7, 11] in special forms. Also, in the case $\phi_\sigma = \phi_\varepsilon$ (2) becomes the Hooke Law:

$$ \sigma(x,t) = \varepsilon(x,t), \quad x \in [0,L], \ t > 0. $$

We note that the constitutive functions or distributions ϕ_σ and ϕ_ε appearing in (2) must be taken in the accordance with the Second Law of Thermodynamics. For example, if we take

$$ \phi_\sigma(\gamma) := a^\gamma, \quad \phi_\varepsilon(\gamma) := b^\gamma, \quad \gamma \in (0,1), \ a, b > 0, \tag{5} $$

then there is a restriction $a \le b$, see [1, 3, 5]. The special case when the constitutive distributions ϕ_σ and ϕ_ε are given by

$$ \phi_\sigma(\gamma) := \delta(\gamma) + a\,\delta(\gamma-\alpha), \quad \phi_\varepsilon(\gamma) := \delta(\gamma) + b\,\delta(\gamma-\alpha), \quad \alpha \in (0,1), \ 0 < a \le b, \tag{6} $$

where δ is the Dirac distribution, is of particular interest. This case gives a generalization of the Zener constitutive equation for a viscoelastic body. The waves in such type of materials were studied in [10]. If ϕ_σ and ϕ_ε are given by

$$ \begin{aligned} \phi_\sigma(\gamma) &:= \delta(\gamma) + a\,\delta(\gamma-\alpha), \\ \phi_\varepsilon(\gamma) &:= b_0\,\delta(\gamma-\beta_0) + b_1\,\delta(\gamma-\beta_1) + b_2\,\delta(\gamma-\beta_2), \end{aligned} \tag{7} $$

where a, b_0, b_1, b_2 are positive constants and $0 < \alpha < \beta_0 < \beta_1 < \beta_2 \le 1$, then one obtains a constitutive equation proposed in [17]. We note that the system (1) - (4), with the choice of constitutive functions and distributions (5) and (6), is considered in [8]. We refer to [12, 14, 15] for the detailed account on the use of fractional calculus in viscoelasticity.

Note that we can apply our results in the study of behavior of solid-like materials, as done in [8] for the cases when M takes the forms (10) and (11). The behavior of a fluid-like material in

a special form is analyzed in [5], where we used constitutive distributions in the form given by (7).

The paper is organized as follows. We present in § 3 main results of our work formulated as Theorems 1 and 2. Proofs of these theorems are given in two steps: the first one is given at the beginning of § 4, while the second one is given in § 4.2. We obtain the displacement u and stress σ as solutions to (1) - (4) in the convolution form by the use of the Laplace transform method. In order to be able to invert the Laplace transform in § 4.2, we need several auxiliary results, that are given in § 4.1. On the basis of Theorems 1 and 2, we discuss, in § 5, Theorems 14 and 15, a model of elastic rod as a special case.

2 Notation and assumptions

In the sequel we consider analytic functions in

$$V=\mathbb{C}\backslash(-\infty,0]=\left\{z=re^{\mathrm{i}\varphi}\mid r>0,\ \varphi\in(-\pi,\pi)\right\}.$$

We often use notation $|s|\to\infty$ and $|s|\to 0$, where we assume that $s\in V$.

The Laplace transform of $f\in L^1_{loc}(\mathbb{R})$, $f\equiv 0$ in $(-\infty,0]$ and $|f(t)|\leq ce^{kt}$, $t>0$, for some $k>0$, is defined by

$$\tilde{f}(s)=\mathcal{L}[f(t)](s):=\int_0^\infty f(t)\,e^{-st}\mathrm{d}t,\quad \operatorname{Re}s>k \tag{8}$$

and analytically continued in an appropriate domain (in our case V).

We consider spaces of tempered distributions supported by $\bar{\mathbb{R}}_+=[0,\infty)$, denoted by $\mathcal{S}'_+$. The Laplace transform of distributions in $\mathcal{S}'_+$ is derived from (8), since tempered distributions are derivatives of polynomially bounded continuous functions. We refer to [18] for the spaces of distributions, as well as for the Laplace and Fourier transforms in such spaces. We use $C\left([0,1],\mathcal{S}'_+\right)$ to denote the space of continuous functions on $[0,1]$ with values in $\mathcal{S}'_+$.

In the analysis that follows we shall need the properties of an analytic function M, defined in appropriate domain $\mathcal{V}\subset\mathbb{C}$

$$M(s):=\sqrt{\frac{\int_0^1\phi_\sigma(\gamma)\,s^\gamma\mathrm{d}\gamma}{\int_0^1\phi_\varepsilon(\gamma)\,s^\gamma\mathrm{d}\gamma}},\quad s\in\mathcal{V}. \tag{9}$$

We shall have $\mathcal{V}=V$. For the cases of constitutive functions (5) and (6), M has the respective forms

$$M(s) = \sqrt{\frac{\ln(bs)}{\ln(as)}\frac{as-1}{bs-1}},\quad s\in V,\ a\leq b, \tag{10}$$

$$M(s) = \sqrt{\frac{1+as^\alpha}{1+bs^\alpha}},\quad s\in V,\ \alpha\in(0,1),\ a\leq b. \tag{11}$$

In our analysis the next function has a special role

$$f(s):=sM(s)\sinh(\kappa sM(s))+\kappa\cosh(\kappa sM(s)),\quad s\in V. \tag{12}$$

As it will be seen from (28) and (29), f is a denominator of functions $\tilde{P}$ and $\tilde{Q}$, which, after the inversion of the Laplace transform, represent solution kernels of u and σ, respectively.

We summarize all the assumptions used throughout the manuscript. Let M be of the form

$$M(s) = r(s) + \mathrm{i}h(s), \quad \text{as} \quad |s| \to \infty.$$

We assume:

(A1)

$$\lim_{|s|\to\infty} r(s) = c_\infty > 0, \quad \lim_{|s|\to\infty} h(s) = 0, \quad \lim_{|s|\to 0} M(s) = c_0,$$

for some constants $c_\infty, c_0 > 0$.

Let $s_n = \xi_n + \mathrm{i}\zeta_n$, $n \in \mathbb{N}$, satisfy the equation

$$f(s) = 0, \quad s \in V, \tag{13}$$

where f is given by (12).

(A2) There exists $n_0 > 0$, such that for $n > n_0$

$$\operatorname{Im} s_n \in \mathbb{R}_+ \Rightarrow h(s_n) \leq 0, \qquad \operatorname{Im} s_n \in \mathbb{R}_- \Rightarrow h(s_n) \geq 0,$$

where $h := \operatorname{Im} M$.

(A3) There exist $s_0 > 0$ and $c > 0$ such that

$$\left| \frac{\mathrm{d}}{\mathrm{d}s}(sM(s)) \right| \geq c, \quad |s| > s_0.$$

(A4) For every $\gamma > 0$ there exists $\theta > 0$ and s_0 such that

$$|(s + \Delta s) M(s + \Delta s) - sM(s)| \leq \gamma, \quad \text{if} \quad |\Delta s| < \theta \quad \text{and} \quad |s| > s_0.$$

Alternatively to (A2), in Proposition 6, we shall consider the following assumption.

(B) $|h(s)| \leq \frac{C}{|s|}$, $|s| > s_0$, for some constants $C > 0$ and $s_0 > 0$.

It is shown in [8] that (A1) - (A4) hold for M given by (10) and (11).

3 Theorems on the existence and uniqueness

Our central results are stated in the next two theorems on the existence, uniqueness and properties of u and σ. Recall, f is given by (12) and s_n, $n \in \mathbb{N}$, are solutions of (13).

Theorem 1 *Let $F \in \mathcal{S}'_+$ and suppose that M satisfies assumptions* (A1) - (A4)*. Then the unique solution u to (1) - (4) is given by*

$$u(x,t) = F(t) * P(x,t), \quad x \in [0,1], \ t > 0, \tag{14}$$

where

$$\begin{aligned} P(x,t) &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa x q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right) + \kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q \\ &\quad + 2\sum_{n=1}^{\infty}\operatorname{Re}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right), \quad x \in [0,1], \ t > 0, \qquad (15) \\ P(x,t) &= 0, \quad x \in [0,1], \ t < 0. \end{aligned}$$

The residues are given by

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)=\left[\frac{1}{s}\frac{M(s)\sinh(\kappa xsM(s))}{\frac{\mathrm{d}}{\mathrm{d}s}f(s)}\mathrm{e}^{st}\right]_{s=s_n},\quad x\in[0,1],\ t>0, \tag{16}$$

Then $P\in C([0,1]\times[0,\infty))$ *and* $u\in C\left([0,1],\mathcal{S}'_+\right)$. *In particular, if* $F\in L^1_{loc}([0,\infty))$, *then* u *is continuous on* $[0,1]\times[0,\infty)$.

The following theorem is related to stress σ. We formulate this theorem with $F=H$, where H denotes the Heaviside function, while the more general cases of F are discussed in Remark 3, below.

Theorem 2 *Let* $F=H$ *and suppose that* M *satisfies assumptions* (A1) - (A4). *Then the unique solution* σ_H *to (1) - (4), is given by*

$$\begin{aligned}\sigma_H(x,t) &= H(t)+\frac{\kappa}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{\cosh\left(\kappa xqM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{qM(q\mathrm{e}^{\mathrm{i}\pi})\sinh(\kappa qM(q\mathrm{e}^{\mathrm{i}\pi}))+\kappa\cosh(\kappa qM(q\mathrm{e}^{\mathrm{i}\pi}))}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q\\ &\quad +2\sum_{n=1}^\infty \operatorname{Re}\left(\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st},s_n\right)\right),\quad x\in[0,1],\ t>0, \qquad (17)\\ \sigma_H(x,t) &= 0,\quad x\in[0,1],\ t<0. \qquad (18)\end{aligned}$$

The residues are given by

$$\operatorname{Res}\left(\sigma_H(x,s)\,\mathrm{e}^{st},s_n\right)=\left[\frac{\kappa\cosh(\kappa xsM(s))}{s\frac{\mathrm{d}}{\mathrm{d}s}f(s)}\mathrm{e}^{st}\right]_{s=s_n},\quad x\in[0,1],\ t>0. \tag{19}$$

In particular, σ_H *is continuous on* $[0,1]\times[0,\infty)$.

Remark 3

1. *The assumption* $F=H$ *in Theorem 2 can be relaxed by requiring that* F *is locally integrable and*
$$\tilde{F}(s)\approx\frac{1}{s^\alpha},\quad as\quad |s|\to\infty,$$
for some $\alpha\in(0,1)$. *This condition ensures the convergence of the series in (17).*

2. *If* $F=\delta$, *or even* $F(t)=\frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta(t)$, *one uses* σ_H, *given by (17), in order to obtain* σ *as the* $k+1$*-th distributional derivative:*
$$\sigma=\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}\sigma_H\in C\left([0,1],\mathcal{S}'_+\right).$$

4 Proofs of Theorems 1 and 2

Theorems 1 and 2 will be proved in two steps.
Step 1. Applying formally the Laplace transform to (1) - (4), we obtain

$$\frac{\partial}{\partial x}\tilde{\sigma}(x,s)=\kappa^2s^2\tilde{u}(x,s),\quad \tilde{\varepsilon}(x,s)=\frac{\partial}{\partial x}\tilde{u}(x,s),\quad x\in[0,1],\ s\in V, \tag{20}$$

$$\tilde{\sigma}(x,s)\int_0^1\phi_\sigma(\gamma)\,s^\gamma\mathrm{d}\gamma=\tilde{\varepsilon}(x,s)\int_0^1\phi_\varepsilon(\gamma)\,s^\gamma\mathrm{d}\gamma,\quad x\in[0,1],\ s\in V, \tag{21}$$

$$\tilde{u}(0,s)=0,\quad \tilde{\sigma}(1,s)+s^2\tilde{u}(1,s)=\tilde{F}(s),\quad s\in V. \tag{22}$$

By (21) and (9), we have

$$\tilde{\sigma}(x,s) = \frac{1}{M^2(s)}\tilde{\varepsilon}(x,s), \quad x \in [0,1], \ s \in V. \tag{23}$$

In order to obtain the displacement u, we use (20) and (23) to obtain

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x,s) - (\kappa s M(s))^2 \tilde{u}(x,s) = 0, \quad x \in [0,1], \ s \in V. \tag{24}$$

The solution of (24) is

$$\tilde{u}(x,s) = C_1(s)\,\mathrm{e}^{\kappa x s M(s)} + C_2(s)\,\mathrm{e}^{-\kappa x s M(s)}, \quad x \in [0,1], \ s \in V,$$

where C_1 and C_2 are arbitrary functions which are determined from $(22)_1$ as $2C = C_1 = -C_2$. Therefore,

$$\tilde{u}(x,s) = C(s)\sinh(\kappa x s M(s)), \quad x \in [0,1], \ s \in V. \tag{25}$$

By (9), $(20)_2$, (23) and (25) we have

$$\tilde{\sigma}(x,s) = C(s)\frac{\kappa s}{M(s)}\cosh(\kappa x s M(s)), \quad x \in [0,1], \ s \in V. \tag{26}$$

Using (25) and (26) at $x = 1$, by $(22)_2$, we obtain

$$C(s) = \frac{M(s)\tilde{F}(s)}{s\left(sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s))\right)}, \quad s \in V.$$

Therefore, the Laplace transforms of displacement (25) and stress (26) are

$$\tilde{u}(x,s) = \tilde{F}(s)\tilde{P}(x,s) \ \text{ and } \ \tilde{\sigma}(x,s) = \tilde{F}(s)\tilde{Q}(x,s), \quad x \in [0,1], \ s \in V, \tag{27}$$

where

$$\tilde{P}(x,s) = \frac{1}{s}\frac{M(s)\sinh(\kappa x s M(s))}{sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s))}, \quad x \in [0,1], \ s \in V, \tag{28}$$

$$\tilde{Q}(x,s) = \frac{\kappa\cosh(\kappa x s M(s))}{sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s))}, \quad x \in [0,1], \ s \in V. \tag{29}$$

Applying the inverse Laplace transform to (27) we obtain u in the form (14), while by using $F = H$, i.e., $\tilde{F}(s) = \frac{1}{s}$, in (27) we obtain σ_H in the form (17), (18).

In the sequel we shall justify the formal calculation given above.

4.1 Auxiliary results

4.1.1 Zeros of the function f

The following propositions establish the location and the multiplicity of poles of functions $\tilde{P}$ and $\tilde{Q}$, given by (28) and (29), respectively.

Proposition 4 *Assume* (A1). *Equation (13) has countably many solutions s_n, $n \in \mathbb{N}$, with the properties*

$$s_n M(s_n) = \mathrm{i} w_n, \quad \tan(\kappa w_n) = \frac{\kappa}{w_n}, \quad w_n \in \mathbb{R}, \ w_n \neq 0. \tag{30}$$

Complex conjugate $\bar{s}_n$ also satisfies (13), $n \in \mathbb{N}$.

Note $\operatorname{Im} s_n \neq 0$ in (30), so all the solutions belong to V.

Proof. We seek for the solutions of (13), or equivalently of

$$\mathrm{e}^{2\kappa s M(s)} = \frac{sM(s) - \kappa}{sM(s) + \kappa}, \quad s \in V. \tag{31}$$

Put $sM(s) = v(s) + \mathrm{i}w(s)$, $s \in V$, where v, w are real-valued functions. Taking the modulus of (31), we obtain

$$\mathrm{e}^{2\kappa v} = \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2}. \tag{32}$$

Fix w and let $v < 0$. Then

$$\mathrm{e}^{2\kappa v} < 1 \text{ and } \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2} > 1.$$

Now let $v > 0$. Then

$$\mathrm{e}^{2\kappa v} > 1 \text{ and } \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2} < 1.$$

Thus, in both cases we have a contradiction and we conclude that the solutions to (32) satisfy $v(s) = 0$. Therefore, the solutions of (31) satisfy

$$sM(s) = \mathrm{i}w(s), \quad s \in V.$$

Inserting this into (13) yields

$$\tan(\kappa w) = \frac{\kappa}{w}, \quad w \in \mathbb{R}. \tag{33}$$

Since the tangent function is periodic, we conclude that there are countably many values of w, denoted by w_n, $n \in \mathbb{N}$, satisfying (33). Hence, we have (30).

In order to prove that the solutions $s_n \in V$ of (13) are complex conjugated we note that $M(\bar{s}) = \overline{M(s)}$, $s \in V$. By (30), $\bar{s}_n M(\bar{s}_n) = \overline{s_n M(s_n)} = -\mathrm{i}w_n$. Thus, $\bar{s}_n$ also solves (13). ∎

Proposition 5 *Assume* (A1)*. Positive (negative) solutions of* $\tan(\kappa w_n) = \frac{\kappa}{w_n}$ *satisfy* $w_n \approx \frac{n\pi}{\kappa}$ *(*$w_n \approx -\frac{n\pi}{\kappa}$*) as* $n \to \infty$.

Proof. As we noted, if w_n satisfies (30), then $-w_n$ also satisfies (30). Since $\frac{\kappa}{w_n}$ monotonically decreases to zero for all $w_n > 0$, by (30), we have that κw_n behave as zeros of the tangent function, i.e., that

$$w_n \approx \frac{n\pi}{\kappa} \quad (w_n \approx -\frac{n\pi}{\kappa}) \text{ as } n \to \infty.$$

∎

Proposition 6 *Assume* (A1), (A2), *or* (A1), (B)*. Then there exist* $\xi_0 > 0$ *and* $n_0 \in \mathbb{N}$ *so that the real part of* s_n, $n \in \mathbb{N}$, *denoted by* ξ_n *satisfies* $\xi_n < \xi_0$, $n > n_0$. *Moreover, if we assume additionally* (A3)*, then the solutions* s_n *of (13) are of multiplicity one for* $n > n_0$.

Proof. Assume (A1) and (A2). By (30) we have

$$(\xi_n + \mathrm{i}\zeta_n)(r(s_n) + \mathrm{i}h(s_n)) \approx \mathrm{i}w_n, \quad n > n_0.$$

This implies

$$\xi_n r(s_n) = \zeta_n h(s_n), \quad n \in \mathbb{N}, \tag{34}$$

$$\xi_n h(s_n) + \zeta_n r(s_n) \approx w_n, \quad n > n_0. \tag{35}$$

Inserting (34) into (35), we obtain

$$\zeta_n \approx \frac{n\pi}{c_\infty \kappa} \text{ because of } w_n \approx \frac{n\pi}{\kappa}, \quad n > n_0, \tag{36}$$

or

$$\zeta_n \approx -\frac{n\pi}{c_\infty \kappa} \text{ because of } w_n \approx -\frac{n\pi}{\kappa}, \quad n > n_0. \tag{37}$$

In the case of (36), we have

$$\xi_n \approx \frac{\zeta_n}{c_\infty} h(s_n) \le 0, \quad n > n_0,$$

since s_n belongs to the upper complex half-plane. In the case of (37), we have

$$\xi_n \approx \frac{\zeta_n}{c_\infty} h(s_n) \le 0, \quad n > n_0,$$

since then s_n belongs to the lower complex half-plane. Thus, in both cases $\xi_n \le 0$ for sufficiently large n. This proves the first assertion.

Assume (A1) and (B). This and (30) imply $s_n \approx \mathrm{i}\frac{w_n}{c_\infty}$, for $n > n_0$. Thus, by (34) and (35), we obtain

$$|\xi_n| \le \frac{|w_n|}{c_\infty^2} \frac{C}{|s_n|} \le \frac{C}{c_\infty}, \quad n > n_0.$$

So, the real parts ξ_n of solutions s_n of (13) satisfy $\xi_n \in \left[-\frac{C}{c_\infty}, \frac{C}{c_\infty}\right]$, for $n > n_0$. This is even a stronger condition for the zeros, but it will not be used in the sequel.

In order to prove that the solutions s_n, $n > n_0$, of f are of multiplicity one, we use (A3), differentiate (13) and obtain

$$\frac{\mathrm{d}f(s)}{\mathrm{d}s} = \left(\left(1+\kappa^2\right)\sinh(\kappa s M(s)) + \kappa s M(s)\cosh(\kappa s M(s))\right) \frac{\mathrm{d}}{\mathrm{d}s}(sM(s)), \quad s \in V.$$

Calculating the previous expression at s_n, we have that

$$\left| \left.\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right|_{s=s_n} \right| = \left|\left(1+\kappa^2\right)\sin(\kappa w_n) + \kappa w_n \cos(\kappa w_n)\right| \left| \left[\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n} \right|$$

is different from zero by (A3). ■

4.1.2 Estimates

In the sequel, we assume (A1) - (A4).

Let $R > 0$. A quarter of a disc, denoted by D, and its boundary Γ are defined by

$$\begin{aligned} D &= D_R = \left\{ s = \rho\,\mathrm{e}^{\mathrm{i}\varphi} \mid \rho \le R,\ \varphi \in \left(\frac{\pi}{2}, \pi\right) \right\}, \\ \Gamma &= \Gamma_R = \left\{ s = R\,\mathrm{e}^{\mathrm{i}\varphi} \mid \varphi \in \left(\frac{\pi}{2}, \pi\right) \right\}. \end{aligned}$$

Let S be the set of all solutions of (13) in D.

In the calculation of P and Q in the next subsections, we shall need the estimates given in the next two lemmas. Recall, S is the set of zeros.

Lemma 7 *Let $\eta > 0$ and*

$$D_\eta = \{s \in D \mid |s - s_j| > \eta,\ s_j \in S\}.$$

Then there exist $s_0 > 0$ and $p_\eta > 0$, such that

$$|f(s)| > p_\eta, \quad \text{if } s \in D_\eta,\ |s| > s_0. \tag{38}$$

Remark 8 *We shall have in the sequel that certain assertions hold for $n > n_0$. This is related to the subindexes of the solutions to (13), but it also implies that we consider domains in D where $|s| > s_0$, where s_0 depends on n_0.*

Proof. If (38) does not hold, then there exists a sequence $\{\tilde{s}_n\}_{n\in\mathbb{N}} \in D_\eta$ such that

$$|f(\tilde{s}_n)| = \eta_n \to 0, \quad n \to \infty. \tag{39}$$

This implies

$$|\mathrm{Re}\,(\tilde{s}_n M(\tilde{s}_n))| \to 0, \qquad |\mathrm{Im}\,(\tilde{s}_n M(\tilde{s}_n))| \to \infty, \quad \text{as } n \to \infty.$$

Our aim is to show that there exist $N \in \mathbb{N}$ and $s_N \in S$, so that

$$|\tilde{s}_N - s_N| \le \eta \tag{40}$$

and this will be the contradiction. Let $\delta < \frac{\pi}{c_\infty \kappa}$ and $\delta \ll \eta$. Recall that there exists n_0, such that for $s_n \in S$, $n > n_0$, there holds

$$\mathrm{Re}\,(s_n M(s_n)) = 0, \quad \left|\mathrm{Im}\,(s_n M(s_n)) - \frac{n\pi}{c_\infty \kappa}\right| < \frac{\delta}{2}.$$

Now consider the intervals

$$I_n = \left(\frac{n\pi}{c_\infty\kappa} + \frac{\delta}{2}, \frac{(n+1)\pi}{c_\infty\kappa} - \frac{\delta}{2}\right), \quad I_{n+1} = \left(\frac{(n+1)\pi}{c_\infty\kappa} + \frac{\delta}{2}, \frac{(n+2)\pi}{c_\infty\kappa} - \frac{\delta}{2}\right), \ldots.$$

Since $|\mathrm{Re}\,(\tilde{s}_n M(\tilde{s}_n))| \to 0$, we see that

$$|\mathrm{Im}\,(\tilde{s}_n M(\tilde{s}_n))| \in I_n \cup I_{n+1} \cup \ldots, \quad n > n_0.$$

Put $\kappa \tilde{s}_n M(\tilde{s}_n) = t_n + \mathrm{i}\tau_n$, $n \in \mathbb{N}$. We have

$$\begin{aligned} 2\kappa f(\tilde{s}_n) \;=\; & t_n\left(\mathrm{e}^{t_n} - \mathrm{e}^{-t_n}\right)\cos\tau_n - \tau_n\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin\tau_n + \kappa^2\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos\tau_n \\ & + \mathrm{i}\left(t_n\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin\tau_n + \tau_n\left(\mathrm{e}^{t_n} - \mathrm{e}^{-t_n}\right)\cos\tau_n + \kappa^2\left(\mathrm{e}^{t_n} - \mathrm{e}^{-t_n}\right)\sin\tau_n\right). \end{aligned} \tag{41}$$

Put $\tau_n = k_n\pi + r_n$, $r_n < \pi$, where $k_n, n \in \mathbb{N}$ is an increasing sequence of natural numbers. We shall show that r_n must have a subsequence tending to zero and this will lead to (40), i.e., to the contradiction with $\tilde{s}_n \in D_\eta$.

So let us assume that r_n does not have a subsequence converging to zero; so $r_n > \frac{\delta}{2}$, $n > n_0$. Having in mind that $t_n \to 0$, and dropping summands tending to zero in (41), one obtains $(n \to \infty)$

$$\begin{aligned} |2\kappa f(\tilde{s}_n)| \;&\sim\; |-\tau_n\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin\tau_n + \kappa^2\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos\tau_n| \\ &=\; |-(k_n\pi + r_n)\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin r_n + \kappa^2\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos r_n| \\ &\sim\; |-k_n\pi\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin r_n - r_n\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin r_n + \kappa^2\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos r_n|. \end{aligned}$$

Now we see that the first summand on the last right hand side tends to infinity, while the second and the third one are bounded. This is in contradiction with (39), so lemma is proved. ■

In the next proposition we shall find estimates on f needed for the later calculation of integrals.

Proposition 9

(*i*) *Let D_0 be a subdomain of D. If $|\mathrm{Re}\,(sM(s))| > d$, $s \in D_0 \subset D$, then there exist $c > 0$ and $s_0 > 0$ such that*

$$|f(s)| \geq c|sM(s)|\,|\sinh(\kappa sM(s))|\,, \quad s \in D_0,\ |s| > s_0.$$

(*ii*) *If $s \in D_\eta$, $|s| > s_0$ (see Lemma 7), then $|f(s)| \geq p_\eta$.*

Note that in the case (*ii*) condition $|\mathrm{Re}\,(sM(s))| \leq d$ is not assumed, although we consider this case in (*ii*), since this part is already a consequence of Lemma 7.

Proof. (*i*) follows from the fact that $sM(s)\sinh(\kappa sM(s))$ tends faster to the infinity than $\cosh(\kappa sM(s))$ when $|s| \to \infty$, $s \in D_0$. So, for some $c > 0$ and $|s| > s_0$,

$$\begin{aligned} |f(s)| &\geq |sM(s)|\,|\sinh(\kappa sM(s))| - \kappa^2\,|\cosh(\kappa sM(s))| \\ &\geq c\,|sM(s)|\,|\sinh(\kappa sM(s))|\,. \end{aligned}$$

This implies the assertion. ■

We need one more estimate of f in the case when we have to control how s is close to the zero set S of f. This is needed for the small deformation of the circle arc Γ_R near the point of Γ_R which is close to some zero of f. We need assumptions (A3) and (A4).

For the later use we choose $\varepsilon > 0$ such that $\varepsilon < \frac{\theta}{2}$ (θ is from (A4)) and that the difference $|s_1M(s_1) - s_2M(s_2)| \leq \gamma$ implies small differences

$$|\cosh(s_1M(s_1)) - \cosh(s_2M(s_2))| < \delta_1,\ |\sinh(s_1M(s_1)) - \sinh(s_2M(s_2))| < \delta_1,$$

as we shall need in the proof of the next lemma (in (48)).

Lemma 10 *Let $0 < \varepsilon < \frac{\theta}{2}$. Then there exist $n_0 \in \mathbb{N}$ and $d > 0$ such that for $s_j \in S$*

$$j > n_0,\ \varepsilon < |s - s_j| \leq 2\varepsilon \Rightarrow \mathrm{Re}\,(sM(s)) > d. \tag{42}$$

Proof. Since for suitable n_0

$$|s_{n+1} - s_n| \geq |\mathrm{Im}\,(s_{n+1} - s_n)| \approx \frac{\pi}{c_\infty \kappa} > \frac{\pi}{2c_\infty \kappa},\ \ n > n_0,$$

we have that the balls $L(s_j, 2\varepsilon)$ are disjoint for $j > n_0$. Let $j > n_0$ and $\varepsilon < |s - s_j| \leq 2\varepsilon$. By the Taylor formula we have

$$|f(s) - f(s_j)| = \left|\frac{\mathrm{d}f(\bar{s})}{\mathrm{d}s}\right| |s - s_j| > \varepsilon \left|\frac{\mathrm{d}f(\bar{s} - s_j)}{\mathrm{d}s}\right|,\ \ \varepsilon < |\bar{s}| \leq 2\varepsilon. \tag{43}$$

So with n_0 large enough we have $|s| > s_0$ so that (A3) implies

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right| \geq c,\ \ \text{for}\ \ |s| > s_0.$$

In the sequel we shall refer to the following set of conditions

$$j > n_0,\ |s| > s_0,\ \varepsilon < |s - s_j| \le 2\varepsilon,\ \ \varepsilon < |\bar{s}| \le 2\varepsilon. \tag{44}$$

Assuming (44), it follows

$$\begin{aligned}\left|\frac{\mathrm{d}f\left(\bar{s}\right)}{\mathrm{d}s}\right| &= \left|\left(1+\kappa^2\right)\sinh\left(\kappa\bar{s}M\left(\bar{s}\right)\right) + \kappa\bar{s}M\left(\bar{s}\right)\cosh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right| \left|\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(sM\left(s\right)\right)\right]_{s=\bar{s}}\right| \\ &\ge c\left(\kappa\left|\bar{s}M\left(\bar{s}\right)\right|\left|\cosh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right| - \left(1+\kappa^2\right)\left|\sinh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|\right). \end{aligned} \tag{45}$$

Now, we estimate $|f|$, assuming (44), and obtain

$$\left|f\left(s\right)\right| \le \left|sM\left(s\right)\right|\left|\sinh\left(\kappa sM\left(s\right)\right)\right| + \kappa\left|\cosh\left(\kappa sM\left(s\right)\right)\right|. \tag{46}$$

Using (45), (46) with (44) in (43), we have

$$\begin{aligned}&\left|sM\left(s\right)\right|\left|\sinh\left(\kappa sM\left(s\right)\right)\right| + \kappa\left|\cosh\left(\kappa sM\left(s\right)\right)\right| \\ &\quad > \varepsilon c\left(\kappa\left|\bar{s}M\left(\bar{s}\right)\right|\left|\cosh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right| - \left(1+\kappa^2\right)\left|\sinh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|\right). \end{aligned} \tag{47}$$

The final part of the proof of the lemma is to show that (47) implies that there exists d such that (42) holds if (44) is satisfied. Contrary to (42), assume that there exist sequences $\tilde{s}_n$, s_{j_n} and $d_n \to 0$ such that

$$\left|\mathrm{Re}\left(\tilde{s}_n\right)M\left(\tilde{s}_n\right)\right| \le d_n,\ \text{ if } \varepsilon < \left|\tilde{s}_n - s_{j_n}\right| \le 2\varepsilon, n > n_0.$$

Since $\tilde{s}_n$, $n > n_0$ satisfies (44), by (47), we would have

$$\begin{aligned}&\left|\tilde{s}_nM\left(\tilde{s}_n\right)\right|\left|\sinh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right| + \kappa\left|\cosh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right| \\ &\quad > \varepsilon c\left(\kappa\left|\bar{s}_nM\left(\bar{s}_n\right)\right|\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right| - \left(1+\kappa^2\right)\left|\sinh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|\right). \end{aligned} \tag{48}$$

The second addend on the right hand side tends to zero, thus we neglect it. Moreover, we note that $\kappa\left|\cosh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right|$ cannot majorize $\varepsilon c\kappa\left|\bar{s}_nM\left(\bar{s}_n\right)\right|\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|$ since the second one tends to infinity while the first one is finite. Thus, in (48), leading terms on both sides are the first ones (with $\tilde{s}_nM\left(\tilde{s}_n\right)$ and $\bar{s}_nM\left(\bar{s}_n\right)$) and we skip the second terms on both sides of (48). It follows, with another $c_0 > 0$,

$$\left|\tilde{s}_nM\left(\tilde{s}_n\right)\right|\left|\sinh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right| \ge c_0\left|\bar{s}_nM\left(\bar{s}_n\right)\right|\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|. \tag{49}$$

Now according to (49) we can choose γ (which then determines θ) and ε so that, with suitable δ_1 and c_1, (48) implies

$$\left|\sinh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right| > c_1\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|,$$

on the domain (44). However, this leads to the contradiction, since $\left|\sinh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right| \to 0$, while $\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|$ is close to one, under the assumptions. This proves the lemma. ■

We note that we can choose η in Lemma 7 to be equal to 2ε of Lemma 10. In this way we obtain that Lemma 10 and Proposition 9 imply:

Proposition 11

(*i*) *There exists* $\varepsilon > 0$ *and* $d > 0$ *such that, for* $\varepsilon < |s - s_n| \le 2\varepsilon$, $s_n \in S$ *and* $|s| > s_0$

$$\left|\mathrm{Re}\left(sM\left(s\right)\right)\right| > d$$

and

$$|f(s)| \ge c\,|sM(s)|\,|\sinh(\kappa sM(s))|\,,\ \ \varepsilon < |s - s_n| \le 2\varepsilon,\ s_n \in S,\ |s| > s_0.$$

(*ii*) *Let* $|s - s_n| > 2\varepsilon$, $s_n \in S$, $|s| > s_0$ *and* d, ε, c *be as in* (*i*).

a) *If* $|\mathrm{Re}\,(sM\,(s))| > d$, *then*

$$|f(s)| \geq c\,|sM(s)|\,|\sinh(\kappa sM(s))|\,.$$

b) *If* $|\mathrm{Re}\,(sM\,(s))| \leq d$, *then*

$$|f(s)| > p_{2\varepsilon}$$

(see the comment before the proof of Proposition 9).

With the notation of the previous proposition, we finally come to corollaries which will be used in the subsequent subsections. We keep the notation from Proposition 11.

Corollary 12

(*i*) *There exists* $\varepsilon > 0$ *and* $C > 0$ *such that for* $\varepsilon < |s - s_n| \leq 2\varepsilon$, $s_n \in S$ *and* $|s| > s_0$,

$$\frac{|sM(s)\sinh\left(\kappa xsM\,(s)\right)|}{|f(s)|} \leq \frac{1}{c}\mathrm{e}^{-\kappa(1-x)\,\mathrm{Re}(sM(s))} \leq C;$$

(*ii*) *for* $|s - s_n| > 2\varepsilon$, $s_n \in S$, $|s| > s_0$

$$\frac{|sM(s)\sinh\left(\kappa xsM\,(s)\right)|}{|f(s)|} \leq \frac{1}{c}\mathrm{e}^{-\kappa(1-x)\,\mathrm{Re}(sM(s))} \leq C,$$

or

$$\frac{|sM(s)\sinh\left(\kappa xsM\,(s)\right)|}{|f(s)|} \leq \frac{d\mathrm{e}^{\kappa xd}\left|1 - \mathrm{e}^{-2\kappa xsM(s)}\right|}{p_{2\varepsilon}} \leq C.$$

Since we also need to estimate $\frac{|\cosh(\kappa xsM(s))|}{|f(s)|}$, we use again Proposition 11. Moreover, we use the fact that in the case $|\mathrm{Re}\,(sM\,(s))| > d$, there exists $c > 0$ such that

$$|\cosh\left(\kappa xsM\,(s)\right)| \leq c\,|\sinh\left(\kappa xsM\,(s)\right)|\,, \quad |s| \to \infty.$$

Corollary 13

(*i*) *There exists* $\varepsilon > 0$ *and* $C > 0$ *such that for* $\varepsilon < |s - s_n| \leq 2\varepsilon$, $s_n \in S$ $|s| > s_0$,

$$\frac{|\cosh\left(\kappa xsM\,(s)\right)|}{|f(s)|} \leq \frac{1}{c \cdot c_\infty}\frac{1}{|s|}\mathrm{e}^{-\kappa(1-x)\,\mathrm{Re}(sM(s))} \leq \frac{C}{|s|};$$

(*ii*) *for* $|s - s_n| > 2\varepsilon$, $s_n \in S$, $|s| > s_0$

$$\frac{|\cosh\left(\kappa xsM\,(s)\right)|}{|f(s)|} \leq \frac{1}{c \cdot c_\infty}\frac{1}{|s|}\mathrm{e}^{-\kappa(1-x)\,\mathrm{Re}(sM(s))} \leq \frac{C}{|s|},$$

or

$$\frac{|\cosh\left(\kappa xsM\,(s)\right)|}{|f(s)|} \leq \frac{\mathrm{e}^{\kappa xd}\left|1 + \mathrm{e}^{-2\kappa xsM(s)}\right|}{p_{2\varepsilon}} \leq C.$$

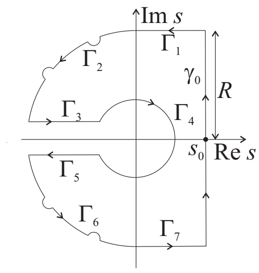

Figure 1: Integration contour Γ

4.2 Continuation of the proofs of Theorems 1 and 2

In this section we finish the proofs of Theorems 1 and 2. First, we finish the proof of Theorem 1. **Proof of Theorem 1. Step 2.** We calculate $P(x,t)$, $x \in [0,1]$, $t \in \mathbb{R}$, by the integration over the contour given in Figure 1. Small, inside or outside half-circles, depending on the zeros of f near Γ_2 and Γ_6, have radius ε determined in Corollaries 12 and 13. This will be explained in the proof. The Cauchy residues theorem yields

$$\oint_{\Gamma} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, \bar{s}_n\right)\right), \tag{50}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \Gamma_7 \cup \gamma_0$, so that poles of $\tilde{P}$ lie inside the contour Γ.

First we show that the series of residues in (15) is real-valued and convergent. Proposition 6 implies that the poles s_n, $n \in \mathbb{N}$, of $\tilde{P}$, given by (28), are simple for sufficiently large n. Then the residues in (50) can be calculated using (16) as

$$\begin{aligned}\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) &= \left[\frac{M(s)\sinh(\kappa x s M(s))}{(1+\kappa^2)\sinh(\kappa s M(s)) + \kappa s M(s)\cosh(\kappa s M(s))}\right]_{s=s_n} \\ &\quad\times\left[\frac{\mathrm{e}^{st}}{s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))}\right]_{s=s_n}, \quad x \in [0,1],\ t > 0.\end{aligned} \tag{51}$$

Substituting (30) in (51), one obtains ($x \in [0,1]$, $t > 0$)

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) = \frac{w_n \sin(\kappa w_n x)}{(1+\kappa^2)\sin(\kappa w_n) + \kappa w_n \cos(\kappa w_n)}\frac{\mathrm{e}^{s_n t}}{\left[s^2\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}}.$$

Proposition 6 implies

$$\left|\mathrm{e}^{s_n t}\right| < \bar{c}\mathrm{e}^{at}, \quad t > 0,$$

for some $a \in \mathbb{R}$. Since $|s_n| \approx \frac{n\pi}{c_\infty \kappa}$ and $|w_n| \approx \frac{n\pi}{\kappa}$, for $n > n_0$, it follows

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le \frac{\bar{c}}{\kappa}\frac{\mathrm{e}^{at}}{\left|\left[s^2\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}\right|}, \quad x \in [0,1],\ t > 0.$$

Now we use assumption (A3). This implies that there exists $K > 0$ such that

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le K\frac{\mathrm{e}^{at}}{n^2}, \quad x \in [0,1],\ t > 0.$$

This implies that the series of residues in (50), i.e., in (15) is convergent.

Now, we calculate the integral over Γ in (50). First, we consider the integral along contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0], \ R > 0\}$, where R is defined as follows. Take n_0 so that $\left|\operatorname{Im} s_n - \frac{n\pi}{c_\infty \kappa}\right| < \eta$, where $0 < \eta \ll \frac{1}{2}\frac{\pi}{c_\infty \kappa}$, for $n > n_0$, and put

$$R = \frac{n\pi}{c_\infty \kappa} + \frac{1}{2}\frac{\pi}{c_\infty \kappa}, \quad n > n_0. \tag{52}$$

By (28) and Corollary 12, we have

$$\left|\tilde{P}(x,s)\right| \leq \frac{C}{|s|^2}, \quad |s| \to \infty. \tag{53}$$

Using (53), we calculate the integral over Γ_1 as

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_1} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{P}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0, \quad x\in[0,1], \ t>0.\end{aligned}$$

Similar arguments are valid for the integral along contour Γ_7. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_7}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0, \quad x\in[0,1], \ t>0.$$

Next, we consider the integral along contour Γ_2. As it is said for Figure 1, the contour Γ_2 consists of parts of contour $\Gamma_R = \left\{s = R\mathrm{e}^{\mathrm{i}\phi} \mid \phi \in \left[\frac{\pi}{2}, \pi\right]\right\}$ and of finite number of contours $\Gamma_\varepsilon = \{|s - s_k| = \varepsilon \mid f(s_k) = 0\}$ encircling the poles s_k, either from inside, or from outside of Γ_R. (Note that the distances between poles are greater than ε, $n > n_0$.) More precisely, if a pole is inside of D_R, then Γ_ε is outside of D_R and if a pole is outside of D_R, then Γ_ε is inside of D_R. By (53), the integral over the contour Γ_2 becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{P}\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0, \quad x\in[0,1], \ t>0,\end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[\frac{\pi}{2}, \pi\right]$. Similar arguments are valid for the integral along Γ_6. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0, \quad x\in[0,1], \ t>0.$$

Consider the integral along Γ_4. Let $|s| \to 0$. Then, by (A1), $sM(s) \to 0$, $\cosh(\kappa s M(s)) \to 1$, $\sinh(\kappa s M(s)) \to 0$ and $\sinh(\kappa x s M(s)) \approx \kappa x s M(s)$. Hence, from (28) we have

$$\left|\tilde{P}(x,s)\right| \approx x\,|M(s)|^2 \approx c_0^2 x, \quad x\in[0,1], \ s\in V, |s|\to 0. \tag{54}$$

The integration along contour Γ_4 gives

$$\begin{aligned}\lim_{r\to 0}\left|\int_{\Gamma_4}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{r\to 0}\int_{-\pi}^{\pi}\left|\tilde{P}\left(x,r\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq c_0^2 x\lim_{r\to 0}\int_{-\pi}^{\pi} r\mathrm{e}^{rt\cos\phi}\mathrm{d}\phi = 0, \quad x\in[0,1], \ t>0,\end{aligned}$$

where we used (54).

Integrals along Γ_3, Γ_5 and γ_0 give $(x \in [0,1]\,,\ t > 0)$

$$\lim_{\substack{R\to\infty\\ r\to 0}} \int_{\Gamma_3} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = \int_0^\infty \frac{M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa x q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q, \tag{55}$$

$$\lim_{\substack{R\to\infty\\ r\to 0}} \int_{\Gamma_5} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = -\int_0^\infty \frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa x q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q, \tag{56}$$

$$\lim_{R\to\infty} \int_{\gamma_0} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}P(x,t)\,. \tag{57}$$

We note that (57) is valid if the inversion of the Laplace transform exists, which is true since all the singularities of $\tilde{P}$ are left from the line γ_0 and the estimates on $\tilde{P}$ over γ_0 imply the convergence of the integral. Summing up (55), (56) and (57) we obtain the left hand side of (50) and finally P in the form given by (15). Analyzing separately

$$\frac{1}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa x q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q,$$
$$2\sum_{n=1}^{\infty}\mathrm{Re}\left(\mathrm{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right),$$

we conclude that both terms appearing in (15) are continuous functions on $t \in [0,\infty)\,,$ for every $x \in [0,1]\,.$ The continuity also holds with respect to $x \in [0,1]$ if we fix $t \in [0,\infty)$. This implies that u is a continuous function on $[0,1]\times[0,\infty)\,.$ From the uniqueness of the Laplace transform it follows that u is unique. Since F belongs to $\mathcal{S}'_+$, it follows that

$$u(x,\cdot) = F(\cdot) * P(x,\cdot) \in \mathcal{S}'_+,$$

for every $x \in [0,1]$ and $u \in C\left([0,1]\,,\mathcal{S}'_+\right).$ Moreover, if $F \in L^1_{loc}([0,\infty))\,,$ then $u \in C([0,1]\times[0,\infty))\,,$ since P is continuous. ■

Next, we complete the proof of Theorem 2.

Proof of Theorem 2. Step 2. Since $\tilde{F}(s) = \tilde{H}(s) = \frac{1}{s}$, $s \neq 0$, by (27) and (29) we obtain

$$\tilde{\sigma}_H(x,s) = \frac{1}{s}\frac{\kappa\cosh(\kappa x s M(s))}{sM(s)\sinh(\kappa s M(s))+\kappa\cosh(\kappa s M(s))},\quad x \in [0,1]\,,\ s \in V. \tag{58}$$

We calculate $\sigma_H(x,t)\,,\ x \in [0,1]\,,\ t \in \mathbb{R}$, by the integration over the same contour from Figure 1. The Cauchy residues theorem yields

$$\oint_{\Gamma} \tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\mathrm{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right)+\mathrm{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, \bar{s}_n\right)\right), \tag{59}$$

so that poles of $\tilde{Q}$ lie inside the contour Γ.

First we show that the series of residues in (17) is convergent and real-valued. The poles s_n, $n \in \mathbb{N}$, of $\tilde{\sigma}_H$, given by (58) are the same as for the function $\tilde{P}$, (28). By Proposition 6 we have

that the poles s_n, $n \in \mathbb{N}$, are simple for sufficiently large n. Then, for $n > n_0$, the residues in (59) can be calculated using (19) as

$$\begin{aligned}\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right) &= \left[\frac{\kappa\cosh(\kappa x s M(s))}{(1+\kappa^2)\sinh(\kappa s M(s)) + \kappa s M(s)\cosh(\kappa s M(s))}\right]_{s=s_n} \\ &\quad\times\left[\frac{\mathrm{e}^{st}}{s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))}\right]_{s=s_n}, \quad x\in[0,1],\ t>0.\end{aligned}$$

By the use of (30) we obtain ($x \in [0,1]$, $t > 0$)

$$\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right) = \frac{\kappa\cos(\kappa w_n x)}{(1+\kappa^2)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)}\frac{\mathrm{e}^{s_n t}}{\left[s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}}.$$

Proposition 6 implies

$$\left|\mathrm{e}^{s_n t}\right| < \bar{c}\mathrm{e}^{at}, \quad t>0,$$

for some $a \in \mathbb{R}$. Since $|s_n| \approx \frac{n\pi}{c_\infty\kappa}$ and $|w_n| \approx \frac{n\pi}{\kappa}$, for $n > n_0$, it follows

$$\left|\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le \frac{\kappa}{n\pi}\frac{\bar{c}\mathrm{e}^{at}}{\left|\left[s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}\right|}, \quad x\in[0,1],\ t>0,\ n>n_0.$$

Now we use assumption (A3) and conclude that there exists $K > 0$ such that

$$\left|\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le K\frac{\mathrm{e}^{at}}{n^2}, \quad x\in[0,1],\ t>0,\ n>n_0.$$

This implies that the series of residues in (59) (i.e., in (17)) is convergent.

Let us calculate the integral over Γ in (59). Consider the integral along contour

$$\Gamma_1 = \{s = p + \mathrm{i}R \mid p\in[0,s_0],\ R>0\},$$

where R is defined by (52). We use estimates

$$\frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \le \frac{C}{|s|}, \quad \text{or} \quad \frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \le C, \tag{60}$$

from Corollary 13 in (58). With the first estimate in (60) we calculate the integral over Γ_1 as

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\le \lim_{R\to\infty}\int_0^{s_0}|\tilde{\sigma}_H(x,p+\mathrm{i}R)|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\le C\kappa\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0, \quad x\in[0,1],\ t>0,\end{aligned}$$

while with the second estimate in (60), we have

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\le \lim_{R\to\infty}\int_0^{s_0}|\tilde{\sigma}_H(x,p+\mathrm{i}R)|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\le C\kappa\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R}\mathrm{e}^{pt}\mathrm{d}p = 0, \quad x\in[0,1],\ t>0,\end{aligned}$$

Similar arguments are valid for the integral along Γ_7. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_7}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0,\ \ x\in[0,1],\ t>0.$$

Next, we consider the integral along contour Γ_2, defined as in the proof of Theorem 1. With the first estimate in (60) we have that the integral over Γ_2 becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{\sigma}_H\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi\\ &\leq C\kappa\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0,\ \ x\in[0,1],\ t>0,\end{aligned}$$

since $\cos\phi\leq 0$ for $\phi\in\left[\frac{\pi}{2},\pi\right]$. The integral over Γ_2, in the case of the second estimate in (60) becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{\sigma}_H\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi\\ &\leq C\kappa\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0,\ \ x\in[0,1],\ t>0,\end{aligned}$$

since $\cos\phi\leq 0$ for $\phi\in\left[\frac{\pi}{2},\pi\right]$. Similar arguments are valid for the integral along Γ_6. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0,\ \ x\in[0,1],\ t>0.$$

Consider the integral along contour Γ_4. Let $|s|\to 0$. Then, by (A1), $sM(s)\to 0$, $\cosh(\kappa sM(s))\to 1$, $\sinh(\kappa sM(s))\to 0$ and $\cosh(\kappa xsM(s))\to 1$. Hence, from (58) we have

$$s\sigma_H(x,s)\approx 1.$$

The integration along contour Γ_4 gives

$$\begin{aligned}\lim_{r\to 0}\int_{\Gamma_4}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= \lim_{r\to 0}\int_{\pi}^{-\pi}\tilde{\sigma}_H\left(x,r\mathrm{e}^{\mathrm{i}\phi}\right)\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\phi}}\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi\\ &= \mathrm{i}\int_{\pi}^{-\pi}\mathrm{d}\phi = -2\pi\mathrm{i},\ \ x\in[0,1],\ t>0. \end{aligned}\tag{61}$$

Integrals along Γ_3, Γ_5 and γ_0 give $(x\in[0,1],\ t>0)$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_3}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s = -\int_0^{\infty}\frac{\kappa\cosh\left(\kappa xqM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q, \tag{62}$$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_5}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s = \int_0^{\infty}\frac{\kappa\cosh\left(\kappa xqM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q, \tag{63}$$

$$\lim_{R\to\infty}\int_{\gamma_0}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sigma_H(x,t). \tag{64}$$

By the same arguments as in the proof of Theorem 1 we have that (64) is valid if the inversion of the Laplace transform exists. This is true since all the singularities of $\tilde{\sigma}_H$ are left from the line γ_0 and appropriate estimates on $\tilde{\sigma}_H$ are satisfied. Adding (61), (62), (63) and (64) we obtain the left hand side of (59) and finally σ_H in the form given by (17).

Function σ_H is a sum of three addends: H and

$$\frac{\kappa}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{\cosh\left(\kappa x q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q,$$

$$2\sum_{n=1}^{\infty}\operatorname{Re}\left(\operatorname{Res}\left(\tilde{\sigma}_H\left(x,s\right)\mathrm{e}^{st},s_n\right)\right).$$

As in the case of Theorem 1, the explicit form of solution implies that σ_H is continuous on $[0,1]\times[0,\infty)$. ■

5 The case of elastic rod

We treat the case of elastic rod separately. Then, for $s\in V$, $M(s)=1$ ($r(s)=1$ and $h(s)\equiv 0$) and clearly, all the conditions (A1) - (A4) hold. By (28) and (29) we have

$$\tilde{P}_{el}(x,s) = \frac{1}{s}\frac{\sinh(\kappa s x)}{s\sinh(\kappa s)+\kappa\cosh(\kappa s)},\quad x\in[0,1],\ s\in V, \tag{65}$$

$$\tilde{Q}_{el}(x,s) = \frac{\kappa\cosh(\kappa s x)}{s\sinh(\kappa s)+\kappa\cosh(\kappa s)},\quad x\in[0,1],\ s\in V. \tag{66}$$

We apply the results of the previous section. Propositions 4 and 5 imply that the zeros of

$$f_{el}(s) := s\sinh(\kappa s)+\kappa\cosh(\kappa s)=0,\quad s\in V,$$

are of the form

$$s_n=\mathrm{i}w_n,\quad \tan(\kappa w_n)=\frac{\kappa}{w_n},\quad w_n\approx\pm\frac{n\pi}{\kappa},\quad \text{as}\quad n\to\infty.$$

Each of these zeros is of multiplicity one for $n>n_0$. Moreover, all the zeros s_n, $n>n_0$, of f_{el} lie on the imaginary axis, so that we do not have the branch point at $s=0$. This implies that the integrals over Γ_3 and Γ_5 (see Figure 1) are equal to zero. So we have the following modifications.

Theorem 14 *Let $F\in\mathcal{S}'_+$. Then the unique solution u to (1) - (4) is given by*

$$u(x,t)=F(t)*P_{el}(x,t),\quad x\in[0,1],\ t>0,$$

where

$$P_{el}(x,t)=2\sum_{n=1}^{\infty}\frac{\sin(\kappa w_n x)\sin(w_n t)}{w_n\left((1+\kappa^2)\sin(\kappa w_n)+\kappa w_n\cos(\kappa w_n)\right)},\quad x\in[0,1],\ t>0. \tag{67}$$

In particular, $u\in C\left([0,1],\mathcal{S}'_+\right)$. Moreover, if $F\in L^1_{loc}([0,\infty))$, then $u\in C([0,1]\times[0,\infty))$.

Proof. The explicit form of P_{el} is obtained from (65) by the use of the Cauchy residues theorem ($x\in[0,1]$, $t>0$)

$$\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{el}}\tilde{P}_{el}(x,s)\mathrm{e}^{st}\mathrm{d}s=\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{P}_{el}(x,s)\mathrm{e}^{st},s_n\right)+\operatorname{Res}\left(\tilde{P}_{el}(x,s)\mathrm{e}^{st},\bar{s}_n\right)\right), \tag{68}$$

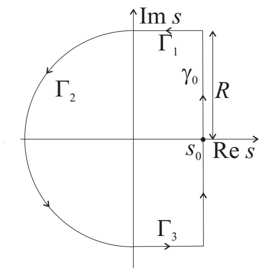

Figure 2: Integration contour Γ_{el}

where the integration contour $\Gamma_{el} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \gamma_0$ is presented in Figure 2.

First we show that the series of residues in (68) is real-valued and convergent. Since the poles $s_n = \mathrm{i}w_n$ (and $\bar{s}_n = -\mathrm{i}w_n$) of $\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}$ are simple for $n > n_0$, the residues in (68) are calculated by ($x \in [0,1]$, $t > 0$)

$$\begin{aligned} \operatorname{Res}\left(\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}, s_n\right) &= \left[\frac{1}{s}\frac{\sinh(\kappa s x)\,\mathrm{e}^{st}}{\frac{\mathrm{d}}{\mathrm{d}s}\left(s\sinh(\kappa s) + \kappa\cosh(\kappa s)\right)}\right]_{s=\mathrm{i}w_n} \\ &= \frac{\sin(\kappa w_n x)\left(\sin(w_n t) - \mathrm{i}\cos(w_n t)\right)}{w_n\left((1+\kappa^2)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)\right)}. \end{aligned}$$

Since $w_n \approx \pm\frac{n\pi}{\kappa}$, as $n \to \infty$, the previous expression becomes

$$\left|\operatorname{Res}\left(\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \leq \frac{k}{n^2}, \quad x \in [0,1],\ t > 0, n > n_0.$$

We conclude that the series of residues is convergent.

Now, we calculate the integral over Γ in (68). First, we consider the integral along contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0],\ R > 0\}$, where R is defined as

$$R = \frac{n\pi}{\kappa} + \frac{1}{2}\frac{\pi}{\kappa}, \quad n > n_0. \tag{69}$$

Let $x \in [0,1]$, $t > 0$. By (65) and Corollary 12, we have

$$\left|\tilde{P}_{el}(x,s)\right| \leq \frac{C}{|s|^2}, \quad |s| \to \infty. \tag{70}$$

Using (70), we calculate the integral over Γ_1 as

$$\begin{aligned} \lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{P}_{el}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0. \end{aligned}$$

Similar arguments are valid for the integral along Γ_3. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_3}\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0, \quad x \in [0,1],\ t > 0.$$

Next, we consider the integral along contour $\Gamma_2 = \left\{ s = R\,\mathrm{e}^{\mathrm{i}\phi} \mid \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}$. By using (70), the integral over the contour Γ_2 becomes

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{\Gamma_2} \tilde{P}_{el}(x,s)\,\mathrm{e}^{st}\mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \tilde{P}_{el}\left(x, R\mathrm{e}^{\mathrm{i}\phi}\right) \right| \left| \mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}} \right| \left| \mathrm{i}R\mathrm{e}^{\mathrm{i}\phi} \right| \mathrm{d}\phi \\
&\leq C \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{R} \mathrm{e}^{Rt\cos\phi} \mathrm{d}\phi = 0, \quad x \in [0,1],\ t > 0,
\end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$. Integrating along the Bromwich contour, we have

$$\lim_{R\to\infty} \int_{\gamma_0} \tilde{P}_{el}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}P_{el}(x,t), \quad x \in [0,1],\ t > 0.$$

Therefore, (68) yields P_{el} in the form (67). The last assertion of the theorem follows from the proof of Theorem 1. ■

Theorem 15 *Let $F = H$. Then the unique solution $\sigma_H^{(el)}$ to (1) - (4), is given by*

$$\sigma_H^{(el)}(x,t) = -2\kappa \sum_{n=1}^{\infty} \frac{\cos(\kappa w_n x)\cos(w_n t)}{w_n\left((1+\kappa^2)\sin(\kappa w_n) + \kappa w_n \cos(\kappa w_n)\right)}, \quad x \in [0,1],\ t > 0. \tag{71}$$

In particular, $\sigma_H^{(el)}$ is continuous on $[0,1] \times [0,\infty)$.

Proof. The explicit forms of $\sigma_H^{(el)}$ is obtained from (66) and $(27)_2$, with $\tilde{F} = \frac{1}{s}$, i.e.

$$\tilde{\sigma}_H^{(el)}(x,s) = \frac{1}{s} \frac{\kappa\cosh(\kappa s x)}{s\sinh(\kappa s) + \kappa\cosh(\kappa s)}, \quad x \in [0,1],\ s \in V,$$

by the use of the Cauchy residues theorem $(x \in [0,1],\ t > 0)$

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{el}} \tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = \sum_{n=1}^{\infty} \left(\mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, s_n \right) + \mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, \bar{s}_n \right) \right), \tag{72}$$

where the integration contour Γ_{el} is the contour from Figure 2.

First we show that the series of residues in (72) is convergent and real-valued. Since the poles $s_n = \mathrm{i}w_n$ (and $\bar{s}_n = -\mathrm{i}w_n$) of $\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}$ are simple for $n > n_0$, the residues in (72) are calculated by

$$\begin{aligned}
\mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, s_n \right) &= \left[\frac{1}{s} \frac{\kappa\cosh(\kappa s x)\,\mathrm{e}^{st}}{\frac{\mathrm{d}}{\mathrm{d}s}\left(s\sinh(\kappa s) + \kappa\cosh(\kappa s)\right)} \right]_{s=\mathrm{i}w_n} \\
&= -\frac{\kappa\cos(\kappa w_n x)\left(\cos(w_n t) + \mathrm{i}\sin(w_n t)\right)}{w_n\left((1+\kappa^2)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)\right)}, \quad n > n_0.
\end{aligned}$$

As $n \to \infty$ $w_n \approx \pm\frac{n\pi}{\kappa}$, so that the previous expression becomes

$$\left| \mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, s_n \right) \right| \leq \frac{k}{n^2}, \quad x \in [0,1],\ t > 0,\ n > n_0.$$

We conclude that the series of residues is convergent.

Let us calculate the integral over Γ_{el} in (72). Consider the integral along contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0], \ R > 0\}$, where R is defined by (69). We use estimates

$$\frac{|\cosh(\kappa x s)|}{|f_{el}(s)|} \leq \frac{C}{|s|}, \quad \text{or} \quad \frac{|\cosh(\kappa x s)|}{|f_{el}(s)|} \leq C, \tag{73}$$

from Corollary 13 in (71). With the first estimate in (73) we calculate the integral over Γ_1 as

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{\Gamma_1} \tilde{\sigma}_H^{(el)}(x,s)\, \mathrm{e}^{st} \mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_0^{s_0} \left| \tilde{\sigma}_H^{(el)}(x, p+\mathrm{i}R) \right| \left| \mathrm{e}^{(p+\mathrm{i}R)t} \right| \mathrm{d}p \\ &\leq C\kappa \lim_{R\to\infty} \int_0^{s_0} \frac{1}{R^2} \mathrm{e}^{pt} \mathrm{d}p = 0, \quad x \in [0,1], \ t > 0, \end{aligned}$$

while with the second estimate in (73), we have

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{\Gamma_1} \tilde{\sigma}_H^{(el)}(x,s)\, \mathrm{e}^{st} \mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_0^{s_0} \left| \tilde{\sigma}_H^{(el)}(x, p+\mathrm{i}R) \right| \left| \mathrm{e}^{(p+\mathrm{i}R)t} \right| \mathrm{d}p \\ &\leq C\kappa \lim_{R\to\infty} \int_0^{s_0} \frac{1}{R} \mathrm{e}^{pt} \mathrm{d}p = 0, \quad x \in [0,1], \ t > 0. \end{aligned}$$

Similar arguments are valid for the integral along Γ_3. Thus, we have

$$\lim_{R\to\infty} \left| \int_{\Gamma_3} \tilde{\sigma}_H^{(el)}(x,s)\, \mathrm{e}^{st} \mathrm{d}s \right| = 0, \quad x \in [0,1], \ t > 0.$$

Next, we consider the integral along contour $\Gamma_2 = \left\{ s = R\,\mathrm{e}^{\mathrm{i}\phi} \mid \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}$. With the first estimate in (73) we have that the integral over Γ_2 becomes

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{\Gamma_2} \tilde{\sigma}_H^{(el)}(x,s)\, \mathrm{e}^{st} \mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \tilde{\sigma}_H^{(el)}\left(x, R\mathrm{e}^{\mathrm{i}\phi}\right) \right| \left| \mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}} \right| \left| \mathrm{i}R\mathrm{e}^{\mathrm{i}\phi} \right| \mathrm{d}\phi \\ &\leq C\kappa \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{R} \mathrm{e}^{Rt\cos\phi} \mathrm{d}\phi = 0, \quad x \in [0,1], \ t > 0, \end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The integral over Γ_2, in the case of the second estimate in (73) becomes

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{\Gamma_2} \tilde{\sigma}_H(x,s)\, \mathrm{e}^{st} \mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \tilde{\sigma}_H\left(x, R\mathrm{e}^{\mathrm{i}\phi}\right) \right| \left| \mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}} \right| \left| \mathrm{i}R\mathrm{e}^{\mathrm{i}\phi} \right| \mathrm{d}\phi \\ &\leq C\kappa \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{Rt\cos\phi} \mathrm{d}\phi = 0, \quad x \in [0,1], \ t > 0, \end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Integrating along the Bromwich contour, we have

$$\lim_{R\to\infty} \int_{\gamma_0} \tilde{\sigma}_H^{(el)}(x,s)\, \mathrm{e}^{st} \mathrm{d}s = 2\pi\mathrm{i}\sigma_H^{(el)}(x,t), \quad x \in [0,1], \ t > 0.$$

Therefore, (72) yields $\sigma_H^{(el)}$ in the form (71). The last assertion of the theorem follows from the proof of Theorem 2. ■

Remark 16 *Numerical analysis for specific choices of constitutive equations, as well as the interpretation of the results is given in [8].*

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References

[1] T.M. Atanackovic, A modified Zener model of a viscoelastic body, Continuum Mech. Thermodyn. 14 (2002) 137–148.

[2] T.M. Atanackovic, A generalized model for the uniaxial isothermal deformation of a viscoelastic body, Acta Mech. 159 (2002) 77–86.

[3] T.M. Atanackovic, On a distributed derivative model of a viscoelastic body, C. R. Mecanique 331 (2003) 687–692.

[4] T.M. Atanackovic, M. Budincevic, S. Pilipovic, On a fractional distributed-order oscillator, J. Phys. A: Math. Gener. 38 (2005) 6703–6713.

[5] T.M. Atanackovic, S. Konjik, Lj. Oparnica, D. Zorica, Thermodynamical restrictions and wave propagation for a class of fractional order viscoelastic rods. Abstract and Applied Analysis 2011 (2011) ID975694, 32p.

[6] T.M. Atanackovic, S. Pilipovic, D. Zorica, Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod. Int. J. Eng. Sci. 49 (2011) 175–190.

[7] T.M. Atanackovic, S. Pilipovic, D. Zorica, Distributed-order fractional wave equation on a finite domain: creep and forced oscillations of a rod. Continuum Mech. Thermodyn. 23 (2011) 305–318.

[8] T.M. Atanackovic, S. Pilipovic, D. Zorica, Forced oscillations of a body attached to a viscoelastic rod of fractional derivative type. Preprint. (2012).

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier B.V, Amsterdam, 2006.

[10] S. Konjik, Lj. Oparnica, D. Zorica, Waves in fractional Zener type viscoelastic media. J. Math. Anal. Appl. 365 (2010) 259–268.

[11] T.T. Hartley, C.F. Lorenzo, Fractional-order system identification based on continuous order-distributions, Signal Process. 83 (2003) 2287–2300.

[12] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London, 2010.

[13] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.

[14] Yu.A. Rossikhin, Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids. Applied Mechanics Reviews, 63 (2010) 010701-1–010701-12.

[15] Yu.A. Rossikhin, M.V. Shitikova, Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. Applied Mechanics Reviews, 63 (2010) 010801-1–010801-52.

[16] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, Amsterdam, 1993.

[17] H. Schiessel, Chr. Friedrich, A. Blumen, Applications to problems in polymer physics and rheology. In *Applications of fractional calculus in physics* (ed R. Hilfer). World Scientific, Singapore, 2000.

[18] V.S. Vladimirov, Equations of Mathematical Physics, Mir Publishers, Moscow, 1984.